

Contents

Zusammenfassung	9
Abstract	11
Introduction	13
1 Fundamentals	23
1.1 Basic Definitions and Vocabulary	23
1.2 Sobolev Spaces	24
1.2.1 Subspaces $H_0^s \subset H^s$	27
1.2.2 Trace Spaces $H^s(\Gamma)$	28
1.2.3 Dual of Sobolev Spaces	30
1.2.4 Regularity Properties	31
1.2.5 Tensor Products of Sobolev Spaces	32
1.3 Besov Spaces	33
1.3.1 Connection to Sobolev Spaces	34
1.4 Elliptic Partial Differential Equations	36
1.4.1 Variational Problems	38
1.4.2 Weak Formulation of Second Order Dirichlet and Neumann Problems	38
1.4.3 Galerkin Method	40
1.5 Nonlinear Elliptic Partial Differential Equations	41
1.5.1 Nemytskij Operators	42
1.5.2 Well-posed Operator Problems	42
1.5.3 Operators of Polynomial Growth	45
2 Multiresolution Analysis and Wavelets	49
2.1 Multiscale Decompositions of Function Spaces	49
2.1.1 Basics	49
2.1.2 Multiresolution Analysis of \mathcal{H}	50
2.1.3 Multiscale Transformation	54
2.1.4 Dual Multiresolution Analysis of \mathcal{H}'	57
2.2 Multiresolutions of L_2 and H^s	60
2.2.1 Approximation and Regularity Properties	61
2.2.2 Norm Equivalences for Sobolev Spaces $H^s \subset L_2$	62
2.2.3 Riesz Stability Properties	63
2.2.4 Operator Representation	64
2.2.5 Preconditioning	65
2.2.6 Riesz Operators for H^s	66
2.3 B-Spline Wavelets on the Interval	70
2.3.1 B-Spline Wavelets	70
2.3.2 Basis Transformations	72
2.4 Multivariate Wavelets	76
2.4.1 Multidimensional Single Scale Basis	77
2.4.2 Anisotropic Tensor-Product Wavelets	77
2.4.3 Isotropic Tensor-Product Wavelets	79

2.5	Full Space Discretizations	84
2.5.1	Best Approximations	84
2.5.2	Stability of the Discretizations	85
3	Adaptive Wavelet Methods based upon Trees	87
3.1	Introduction	87
3.1.1	The Why of Adaptive Wavelet Methods	87
3.1.2	The How of Adaptive Wavelet Methods	88
3.2	Nonlinear Wavelet Approximation	91
3.2.1	Tree Structured Index Sets	91
3.2.2	The Best (Tree) N -Term Approximation	94
3.3	Algorithms for Tree Structured Index Sets	96
3.3.1	The Adaptive Fast Wavelet Transform	96
3.3.2	Tree Coarsening	100
3.3.3	Tree Prediction	104
3.3.4	Approximating the Influence Set	109
3.4	Application of Semilinear Elliptic Operators	115
3.4.1	Adaptive Polynomial Representation	115
3.4.2	Transformation to Local Polynomial Bases	118
3.4.3	Adaptive Nonlinear Operator Application	120
3.4.4	Reference Element Operator Applications	122
3.4.5	Reconstruction of Target Wavelet Indices	124
3.4.6	The Nonlinear Apply Scheme	125
3.5	Application of Linear Operators	132
3.5.1	Evaluation Algorithms for Linear Operators	133
3.5.2	Bilinear Forms	143
3.5.3	Inverses of Linear Operators	147
3.5.4	A Square Weighted Mass Matrix	149
3.6	Trace Operators	151
3.6.1	Trace Operators Parallel to the Coordinate Axes	152
3.7	Anisotropic Adaptive Wavelet Methods	156
3.7.1	A Tree Structure for Anisotropic Wavelet Indices	156
3.7.2	Conversion Algorithms	157
4	Numerics of Adaptive Wavelet Methods	161
4.1	Iterative Solvers	161
4.1.1	The General Scheme	162
4.1.2	The Right Hand Side	163
4.1.3	The Residual	169
4.1.4	Convergence Rates	170
4.2	Richardson Iteration	176
4.2.1	Linear Operators	179
4.2.2	Semilinear Operators	179
4.3	Gradient Iteration	180
4.4	Newton's Method	181
4.4.1	Semilinear Operators	183
4.4.2	Solving the Inner System	184

4.5	Implementational Details	187
4.5.1	The Maximum Level	187
4.5.2	Starting Solver	188
4.5.3	Increasing the Decay Parameter γ	188
4.5.4	Caching Data	188
4.5.5	Zero Subtrees in Differences of Vectors	188
4.5.6	Estimating The Constants	189
4.5.7	About Runtimes	189
4.6	A 2D Example Problem	191
4.6.1	Solving with Richardson Iteration	193
4.6.2	Solving with Gradient Iteration	195
4.6.3	Solving with Newton Iteration	196
4.6.4	Conclusions	197
4.7	A 3D Example Problem	210
4.7.1	Solving with Richardson Iteration	210
4.7.2	Solving with Gradient Iteration	212
4.7.3	Solving with Newton Iteration	212
4.7.4	Conclusions	212
5	Boundary Value Problems as Saddle Point Problems	221
5.1	Saddle Point Problems	221
5.1.1	The Linear Case	221
5.1.2	The Semilinear Case	224
5.2	PDE Based Boundary Value Problems	228
5.2.1	The Fictitious Domain–Lagrange Multiplier Approach	229
5.2.2	The Case $\Omega = \square, \Gamma = $	231
5.2.3	Wavelet Discretization	233
5.3	Adaptive Solution Methods	234
5.3.1	The Normalized Equation	234
5.3.2	A Positive Definite System	235
5.3.3	Uzawa Algorithms	235
5.3.4	Convergence Properties – The Linear Case	239
5.4	A 2D Linear Boundary Value Example Problem	242
5.4.1	Numerical Results	243
5.5	A 2D Linear PDE and a Boundary on a Circle	250
5.5.1	Application of the Trace Operator in Wavelet Coordinates	251
5.5.2	Numerical Results	252
5.6	A 2D Nonlinear Boundary Value Example Problem	255
5.6.1	Numerical Results	255
5.7	A 3D Nonlinear Boundary Value Example Problem	262
5.7.1	Numerical Results	262
6	Résumé and Outlook	269
6.1	Conclusions	269
6.2	Future Work	270
6.2.1	Control Problems Governed by Nonlinear Elliptic PDEs	270
6.2.2	Parallelization Strategies	270

6.2.3	Parabolic Partial Differential Equations	271
A	Wavelet Details	275
A.1	Boundary Adapted Wavelets on the Interval $(0, 1)$	276
A.1.1	Boundary Adapted Hat Functions $d = 2$	276
A.1.2	Boundary Adapted Dual Generators $\tilde{d} = 2$	278
A.1.3	Boundary Adapted Dual Generators $\tilde{d} = 4$	280
A.1.4	Boundary Adapted Wavelets $d = 2, \tilde{d} = 2, j_0 = 3$ (DKU)	282
A.1.5	Boundary Adapted Wavelets $d = 2, \tilde{d} = 4, j_0 = 3$ (DKU)	284
A.1.6	Boundary Adapted Wavelets $d = 2, \tilde{d} = 4, j_0 = 4$ (Primbs)	286
A.2	Wavelet Diagrams	288
A.2.1	1D Diagram	288
A.2.2	2D Diagrams	288
A.2.3	3D Diagrams	289
A.3	Local Polynomial Bases	290
A.3.1	Standard Monomials	290
A.3.2	Normalized Monomials	291
A.3.3	Normalized Shape Functions	293
B	Implementational Details	299
B.1	Compilers and Computers	299
B.1.1	A Few Remarks about CPUs	299
B.1.2	About Compilers	300
B.2	Code Design Rationale	301
B.2.1	Library CMake Options	303
B.2.2	Tensor Products	303
B.2.3	One-dimensional Wavelet Implementations	304
B.2.4	Level-wise Storage in Unordered Containers	305
B.3	Adaptive Storage Containers with Amortized $\mathcal{O}(1)$ Random Access	305
B.3.1	Hash Functions	306
B.3.2	Optimization Strategies	308
C	Notation	313
C.1	General Notation	313
C.2	Special Mathematical Symbols	315
C.3	Spaces	320
C.4	Function Spaces	321
	References	323